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1993 J. Phys. A: Math. Gen. 26 1313

(http://iopscience.iop.org/0305-4470/26/6/014)

J. Phys. A: Math. Gen. 26 (1993) 1313–1316. Printed in the UK

SU(1,1)-invariant solution of the quantum Yang–Baxter equation

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Received 8 June 1992

Abstract. A new approach to the solution of the quantum Yang–Baxter equation is presented and complete SU(1,1)-invariant factorized unitary scattering matrix is constructed.

Completely integrable lattice models are very useful for many problems of modern physics. This is the reason why the problem of constructing solutions of the quantum Yang–Baxter equation (QYBE)—factorized scattering matrices—is widely discussed in scientific literature [1–5]. But a straightforward solution of QYBE for infinite-dimensional representations of some groups (for instance unitary representation of the uncompact Lie group), and thus constructions of exact solutions of completely integrable models with symmetry of this group, is not possible.

In this paper we present a new approach to the solution of QYBE and construct SU(2) and SU(1,1)-invariant factorized unitary scattering matrices as an example.

In accordance with the conception of the universal scattering matrix $\mathcal{R}$ introduced by Drinfeld [6], the QYBE holds for arbitrary representations of the group S, S' and S''

$$\mathcal{R}_{12}^{(SS')}(\lambda)\mathcal{R}_{13}^{(SS'')}(\lambda+\mu)\mathcal{R}_{23}^{(S'S'')}(\mu)=\mathcal{R}_{23}^{(S'S'')}(\mu)\mathcal{R}_{13}^{(SS'')}(\lambda+\mu)\mathcal{R}_{12}^{(SS')}(\lambda) \qquad (1)$$

where λ and μ are rapidities.

Consider S as a representation of the spin-$\frac{1}{2}$. Using the known [7] scattering matrices of a spin-$\frac{1}{2}$ particle by the particle, associated with the representation of SU(2) or SU(1,1) group, we can consider (1) as a linear (finite- or infinite-dimensional) equation on $\mathcal{R}^{(S'S'')}(\mu)$. The scattering matrix $\mathcal{R}^{(\frac{1}{2}S')}$ can be written as

$$\mathcal{R}^{(\frac{1}{2}S')}(\lambda)=X_{ij}\otimes T_{ij}^{(S')}(\lambda) \qquad (2)$$

where X_{ij} are the Hubbard operators ($X_{ij}X_{kl}=\delta_{jk}X_{il}$) in the space of the spin-$\frac{1}{2}$ ($i,j=1,2$) and $T_{ij}^{S'}(\lambda)$ are the operators in the space of the spin S' representation $-V_{S'}$.

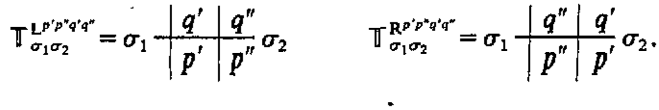
Using (2) we can rewrite (1) in the channel $\frac{1}{2}\otimes S'\otimes S''$ as

$$\mathbb{T}^{\mathrm{L}}\mathcal{R}^{(S'S'')}(\mu)=\mathcal{R}^{(S'S'')}(\mu)\mathbb{T}^{\mathrm{R}} \qquad (3)$$

where $\mathbb{T}^{\mathrm{L}}$ and $\mathbb{T}^{\mathrm{R}}$ are the operators in the space $V_{S'}\otimes V_{S''}$ which are

$$\begin{aligned}\mathbb{T}^{\mathrm{L}}&=T_{\sigma_1 l}^{(S')}(\lambda)\otimes T_{l\sigma_2}^{(S'')}(\lambda+\mu)\\ \mathbb{T}^{\mathrm{R}}&=T_{l\sigma_2}^{(S')}(\lambda)\otimes T_{\sigma_1 l}^{(S'')}(\lambda+\mu).\end{aligned} \qquad (4)$$

Graphically they can be represented as



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Note that $\mathbb{T}^{\mathrm{L}}$ and $\mathbb{T}^{\mathrm{R}}$ represent two possible ways of scattering of the probe spin-$\frac{1}{2}$ particle on the pair of particles belonging to the representations of S', S'' multiplets. It is easy to see that

$$\mathrm{Tr}_\sigma\, \mathbb{T}^{\mathrm{L}} = \mathrm{Tr}_\sigma\, \mathbb{T}^{\mathrm{R}}. \tag{5}$$

We can consider space $V_{S'} \otimes V_{S''}$ as a representation space of two two-node lattice quantum integrable models with $\mathbb{T}^{\mathrm{L}}_{ij}$ and $\mathbb{T}^{\mathrm{R}}_{ij}$—quantum scattering data of these two models with the same commutation relations.

Now let us consider the Bethe states of these models; diagonalizing operators $\mathrm{Tr}\, \mathbb{T}^{\mathrm{L}}$, $\mathrm{Tr}\, \mathbb{T}^{\mathrm{R}}$

$$\begin{aligned} |\lambda_1, \ldots, \lambda_n\rangle_{\mathrm{L}} &= \mathbb{T}^{\mathrm{L}}_{12}(\lambda_1) \ldots \mathbb{T}^{\mathrm{L}}_{12}(\lambda_n)|\Omega\rangle \\ |\lambda_1, \ldots, \lambda_n\rangle_{\mathrm{R}} &= \mathbb{T}^{\mathrm{R}}_{12}(\lambda_1) \ldots \mathbb{T}^{\mathrm{R}}_{12}(\lambda_n)|\Omega\rangle \end{aligned} \tag{6}$$

where $|\Omega\rangle$ is a pseudo-vacuum state.

Since operators $\mathrm{Tr}\, \mathbb{T}$ are diagonalized in the same Bethe states we can see from (5) that the Bethe states of these two models coincide. Norms of the Bethe states are determined only by commutation relations of operators $\mathbb{T}_{ij}(\lambda)$ and eigenvalues of operators $\mathbb{T}_{11}(\lambda)$ and $\mathbb{T}_{22}(\lambda)$ on the vacuum state—which are the same in the considered models. Thus, the only difference between these two models' Bethe states results from a phase factor

$$|\lambda_1, \ldots, \lambda_n\rangle_{\mathrm{L}} = \mathrm{e}^{\mathrm{i}\varphi(\lambda_1, \ldots, \lambda_n)}|\lambda_1, \ldots, \lambda_n\rangle_{\mathrm{R}} \tag{7}$$

where $\mathrm{e}^{\mathrm{i}\varphi(\lambda_1, \ldots, \lambda_n)} \equiv \varphi_n$, which we shall refer to as a dephasing factor.

Note that in accordance with (3) scattering matrix $\mathcal{R}$ (considered as an operator) commutes with the conservation law (5), thus the operator $\mathcal{R}$ can be diagonalized on the Bethe states. Acting on (7) with operator $\mathcal{R}$ and using (3) we have

$$\mathcal{R}|\lambda_1, \ldots, \lambda_n\rangle_{\mathrm{L}} = \mathrm{e}^{\mathrm{i}\varphi(\lambda_1, \ldots, \lambda_n)}\mathbb{T}^{\mathrm{L}}_{12}(\lambda_1) \ldots \mathbb{T}^{\mathrm{L}}_{12}(\lambda_n)\mathcal{R}|\Omega\rangle.$$

We can determine the scattering matrix with an accuracy up to multiplication on an arbitrary function. Here we shall set

$$\mathcal{R}|\Omega\rangle = |\Omega\rangle.$$

Thus

$$\mathcal{R}|\lambda_1, \ldots, \lambda_n\rangle_{\mathrm{L}} = \mathrm{e}^{\mathrm{i}\varphi(\lambda_1, \ldots, \lambda_n)}|\lambda_1, \ldots, \lambda_n\rangle_{\mathrm{L}}. \tag{8}$$

Equation (8) determines the scattering matrix through the Bethe states and dephasing factor. But, in the case of rational scattering matrices the problem simplifies significantly. In this scattering matrices the problem simplifies significantly. In this case the scattering matrix $\mathcal{R}$ can be represented as a direct sum of irreducible representations:

$$\mathcal{R}^{(S'S'')}(\mu) = \sum \varphi_i(\mu)\mathcal{P}_i \tag{9}$$

where $\mathcal{P}_i$ is a projection operator on S_i irreducible representation and $\varphi_i(\mu)$ is a dephasing factor which is the same for all states of this representation.

Now we shall calculate the dephasing factor. Let us rewrite matrices $\mathbb{T}^{\mathrm{L}}$ and $\mathbb{T}^{\mathrm{R}}$ in terms of local operators $T^{(S')}_{ij}$, $T^{(S'')}_{ij}$:

$$\begin{aligned} \mathbb{T}^{\mathrm{L}}_{12}(\lambda) &= T^{(S')}_{11}(\lambda) \otimes T^{(S'')}_{12}(\lambda+\mu) + T^{(S')}_{12}(\lambda) \otimes T^{(S'')}_{22}(\lambda+\mu) \\ \mathbb{T}^{\mathrm{R}}_{12}(\lambda) &= T^{(S')}_{12}(\lambda) \otimes T^{(S'')}_{11}(\lambda+\mu) + T^{(S')}_{22}(\lambda) \otimes T^{(S'')}_{12}(\lambda+\mu). \end{aligned} \tag{10}$$

Now let us consider the n-particle Bethe state (6). This state can be represented as a linear combination of states with a fixed number of excitations in each node (S' and S'') provided that the total number of excitations at both nodes is n. The state which has 0 excitations in node S' and n excitations in node S'' can be written in accordance with (10):

$$|0, n\rangle^{\mathrm{L}} = T_{11}^{(S')}(\lambda_1) \ldots T_{11}^{(S')}(\lambda_n) \otimes T_{12}^{(S'')}(\lambda_1+\mu) \ldots T_{12}^{(S'')}(\lambda_n+\mu)|\Omega\rangle$$
$$|0, n\rangle^{\mathrm{R}} = T_{22}^{(S')}(\lambda_1) \ldots T_{22}^{(S')}(\lambda_n) \otimes T_{12}^{(S'')}(\lambda_1+\mu) \ldots T_{12}^{(S'')}(\lambda+\mu)|\Omega\rangle.$$

The vacuam state is an eigenstate of operators T_{11} and T_{22}, thus

$$\varphi_n = \prod_{i=1}^{n} \frac{a_0'(\lambda_i)}{d_0'(\lambda_i)} \tag{11}$$

where $a_0'(\lambda)$ and $d_0'(\lambda)$ are eigenvalues of operators $T_1^{(S')}(\lambda)$ and $T_{22}^{(S')}(\lambda)$, respectively [7].

In our case of SU(2) or SU(1,1) symmetries, the T matrix can be represented as [7]

$$\begin{aligned} T_{11}^{(S)} &= 1 - \frac{\mathrm{i}\Gamma}{\lambda+\mathrm{i}\Gamma/2}\hat{S}_z \qquad & T_{22}^{(S)} &= 1 + \frac{\mathrm{i}\Gamma}{\lambda+\mathrm{i}\Gamma/2}\hat{S}_z \\ T_{12}^{(S)} &= \frac{\mathrm{i}\Gamma}{\lambda+\mathrm{i}\Gamma/2}\hat{S}_+ \qquad & T_{21}^{(S)} &= \frac{\mathrm{i}\Gamma}{\lambda+\mathrm{i}\Gamma/2}\hat{S}_- \end{aligned} \tag{12}$$

where $\hat{S}_i$ are generators of representation of the group SU(2) or SU(1,1) (spin operators), and $\mathrm{i}\Gamma$ is treated as the Plank constant. We shall consider the pseudo-vacuum state as a state with the lowest possible eigenvalue of spin projection S_z: $\hat{S}_-|\Omega\rangle = 0$; $\hat{S}_z|\Omega\rangle = S_z|\Omega\rangle$ (S_z is a negative number for the case of the SU(2) group, and positive for SU(1,1)).

From (11) and (12)

$$\varphi_n = \prod_{i=1}^{n} \frac{\lambda_i + \mathrm{i}\Gamma/2 - \mathrm{i}\Gamma S_z'}{\lambda_i + \mathrm{i}\Gamma/2 + \mathrm{i}\Gamma S_z'}. \tag{13}$$

Thus we have found the dephasing factor in terms of Bethe's rapidities only. It will be convenient to rewrite (13) as

$$\varphi_n = \frac{\sum_{j=0}^{n} E_j(\mathrm{i}\Gamma/2 - \mathrm{i}\Gamma S_z')^j}{\sum_{j=0}^{n} E_j(\mathrm{i}\Gamma/2 + \mathrm{i}\Gamma S_z')^j} \tag{14}$$

where $E_j(\lambda_1, \ldots, \lambda_n)$ are symmetrical combinations of the Bethe's rapidities:

$$E_n = 1 \qquad E_{n-1} = \lambda_1 + \ldots + \lambda_n \qquad E_{n-2} = \sum_{i \neq j} \lambda_i \lambda_j, \ldots \qquad E_0 = \lambda_1 \ldots \lambda_n.$$

To find Bethe's rapidities let us consider the conservation law of our system. From (4)

$$\begin{aligned} \mathbb{T}_{11}^{\mathrm{L}}(\lambda) &= T_{11}^{(S')}(\lambda) \otimes T_{11}^{(S'')}(\lambda+\mu) + T_{12}^{(S')}(\lambda) \otimes T_{21}^{(S'')}(\lambda+\mu) \\ \mathbb{T}_{22}^{\mathrm{L}}(\lambda) &= T_{22}^{(S')}(\lambda) \otimes T_{22}^{(S'')}(\lambda+\mu) + T_{21}^{(S')}(\lambda) \otimes T_{12}^{(S'')}(\lambda+\mu). \end{aligned} \tag{15}$$

Using (12) we obtain

$$\mathbb{T}_{11}^{\mathrm{L}} + \mathbb{T}_{22}^{\mathrm{L}} = 2 + \frac{(\mathrm{i}\Gamma)^2}{(\lambda+\mathrm{i}\Gamma/2)(\lambda+\mu+\mathrm{i}\Gamma/2)}(\boldsymbol{S}' \cdot \boldsymbol{S}'')$$

where $(\boldsymbol{S}' \cdot \boldsymbol{S}'')$ denotes the scalar product.

Note that a Bethe state has fixed values of both spin and spin projection. Thus, acting on the Bethe state with operators $\mathbb{T}^{L}_{11}+\mathbb{T}^{L}_{22}$ we have

$$(\mathbb{T}^{L}_{11}+\mathbb{T}^{L}_{22})|\lambda_1,\ldots,\lambda_n\rangle=\left(2+\frac{(\mathrm{i}\Gamma)^2}{(\lambda+\mathrm{i}\Gamma/2)(\lambda+\mu+\mathrm{i}\Gamma/2)}\,\alpha\right)|\lambda_1,\ldots,\lambda_n\rangle \tag{16}$$

where α is the eigenvalue of the operator $(\boldsymbol{S}'\cdot\boldsymbol{S}'')$.

On the other hand, using commutation relations we deduce

$$\begin{aligned}\mathbb{T}^{L}_{11}|\lambda_1,\ldots,\lambda_n\rangle&=\prod_{j=1}^{n}\frac{\mathrm{i}\Gamma+\lambda_j-\lambda}{\lambda_j-\lambda}\,a_0'(\lambda)a_0''(\lambda+\mu)|\lambda_1,\ldots,\lambda_n\rangle\\ \mathbb{T}^{L}_{22}|\lambda_1,\ldots,\lambda_n\rangle&=\prod_{j=1}^{n}\frac{\mathrm{i}\Gamma+\lambda-\lambda_j}{\lambda-\lambda_j}\,d_0'(\lambda)d_0''(\lambda+\mu)|\lambda_1,\ldots,\lambda_n\rangle.\end{aligned} \tag{17}$$

Expanding eigenvalues in (16) and (17) in λ and comparing coefficients at the same powers of λ we obtain the following equations which determine quantities E_i $(i=0,\ldots,n)$:

$$\begin{aligned}E_{n-k}=(\mathrm{i}\Gamma)^{-2}[k(2n-k-1)-2k(S_z'+S_z'')]^{-1}\\ \times\sum_{i=n-k+1}^{n}E_i\{C_i^{i-n+k+2}((\mathrm{i}\Gamma)^{i-n+k+2}+(-\mathrm{i}\Gamma)^{i-n+k+2})\\ -C_i^{i-n+k+1}((\mathrm{i}\Gamma)^{i-n+k+1}(\mu+\mathrm{i}\Gamma(1-S_z'-S_z''))\\ +(-\mathrm{i}\Gamma)^{i-n+k+1}(\mu+\mathrm{i}\Gamma(1+S_z'+S_z'')))\\ +C_i^{i-n+k}((\mathrm{i}\Gamma)^{i-n+k}(\mu+\mathrm{i}\Gamma/2-\mathrm{i}\Gamma S_z'')(\mathrm{i}\Gamma/2-\mathrm{i}\Gamma S_z')\\ +(-\mathrm{i}\Gamma)^{i-n+k}(\mu+\mathrm{i}\Gamma/2+\mathrm{i}\Gamma S_z'')(\mathrm{i}\Gamma/2+\mathrm{i}\Gamma S_z'))\}\end{aligned} \tag{18}$$

where C_n^m are binomial coefficients, and consistency condition

$$\alpha=n(n-1)+2n(S_z'+S_z'')+2S_z'S_z''$$

fixes the eigenvalue of spin in the Bethe states.

The solution of equations (14) and (18) is

$$\varphi_n(\mu)=\varphi_{n-1}(\mu)\,\frac{\mu+\mathrm{i}\Gamma(S_z'+S_z''+n-1)}{\mu-\mathrm{i}\Gamma(S_z'+S_z''+n-1)} \tag{19}$$

with $\varphi_0=1$.

Expression (19) coincides with the well known [8] SU(2)-invariant scattering matrix solution. The only difference is the following: S_z', S_z'' are negative integer (or half-integer) numbers in the case of SU(2) group, and any positive integer in the case of SU(1,1).

Thus equations (9) and (19) completely solve our problems and present scattering matrices for all representations of the SU(1,1) group.

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